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Globally Hypoelliptic Systems of Vector Fields*

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This work studies the global hypoellipticity of differential operators associated to certain locally integrable structures defined over compact manifolds. It extends a former result due to Greenfield and Wallach on global hypoellipticity of vector fields defined on the two dimensional torus as well as it makes a connection with results of local hypoellipticity of overdetermined systems of vector fields due to H. M. Maire.

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INTRODUCTION

A linear partial differential operator $\mathbf{P}: C^\infty(\Omega, \mathbf{E}) \to C^\infty(\Omega, \mathbf{E}')$ acting between sections of vector bundles $\mathbf{E}$, $\mathbf{E}'$ over a smooth manifold Ω is said

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to be globally hypoelliptic if the conditions $u \in \mathscr{D}'(\Omega, \mathbf{E})$, $\mathbf{P}u \in C^{\infty}(\Omega, \mathbf{E}')$ imply $u \in C^{\infty}(\Omega, \mathbf{E})$. This is the concept we deal with in this work. We study global hypoellipticity for a certain class of differential operators defined on compact manifolds and associated to locally integrable structures.

The roots of this study lie in a result due to Greenfield and Wallach [G–W] concerning globally hypoelliptic constant coefficients differential operators defined on the two dimensional torus $S^1 \times S^1$. If in the latter the angular variables are denoted by (t, x) and if $a \in \mathbf{R}$ it is proved, in particular, that the operator $\partial_t + a\partial_x$ acting on $C^{\infty}(S^1 \times S^1)$ is globally hypoelliptic if and only a is irrational and not a Liouville number.

We will extend this result as well as make a connection between it and results of local hypoellipticity for overdetermined systems of vector fields due to H. M. Maire [Ma]. To state the problem more precisely, let us denote by M a compact, connected real analytic manifold of dimension $\geqslant 1$. Given a complex, real-analytic closed 1-form ω defined on M we define a differential operator $\mathbf{L}: C^{\infty}(M \times S^1) \to \Lambda^1 C^{\infty}(M \times S^1)$ by

$$\mathbf{L} = d_t + \omega(t) \wedge \partial_x,$$

where d_t is the exterior derivative on M and ∂_x is the derivative with respect to $x \in S^1$ (see Section 1 for the relation between $\mathbf{L}$ and a locally integrable structure on $M \times S^1$ associated to ω).

In Section 2 we assume ω real. In this case we allow ω and M to be just C^{∞} and prove that $\mathbf{L}$ is globally hypoelliptic if and only if ω is neither rational nor Liouville (see Definition 2.1). What is involved here is a property of simultaneous diophantine approximation, similar to what occurred in a work of Moser [M2] on the conjugacy of a family of diffeomorphisms of S^1 to rotations. If $M = S^1$ we recover the Greenfield–Wallach result alluded to above.

In Sections 3, 4, and 5 we assume that $\Im\omega$ is not identically zero. In this case the analysis of the set $\Sigma = \{t \in M : \Im\omega(t) = 0\}$ plays an important role since it is a proper analytic subset of M carrying the projection of the only possible singularities of $u \in \mathscr{D}'(M \times S^1)$ satisfying $\mathbf{L}u \in \Lambda^1 C^{\infty}(M \times S^1)$. Exploiting the geometric properties of Σ we show, in Section 3, that $\Im\omega$ has a primitive ϕ defined in a full neighborhood of Σ and vanishing on it. Once ϕ is available we discuss the global hypoellipticity of $\mathbf{L}$ in Sections 4 and 5.

Maire [Ma] proved that $\mathbf{L}$ is (locally) hypoelliptic (a statement stronger than saying that $\mathbf{L}$ is globally hypoelliptic) provided the local primitives of $\Im\omega$ are open maps. In Section 4, by using the same techniques (integration along the gradient flow of ϕ), we make Maire's result more precise (point hypoellipticity) and also obtain a result of propagation of singularities for the solutions of $\mathbf{L}u = f \in \Lambda^1 C^{\infty}(M \times S^1)$. We prove, in particular, that if each component of Σ has a point at which ϕ is an open map then $\mathbf{L}$ is globally hypoelliptic.

Finally in Section 5 we turn our attention to necessary conditions for the global hypoellipticity of $\mathbf{L}$ and then obtain complete descriptions in two cases: when the cohomological class of $\Re\omega$ vanishes near Σ (Corollary 5.2) and when Σ is free of singularities (Corollary 5.4). This last result shows, in particular, that it is the approximation property of the pullback of $\Re\omega$ to Σ by rational forms which can yield global hypoellipticity for a non locally hypoelliptic $\mathbf{L}$.

1. Preliminary Results

In this work M denotes a compact, connected real-analytic manifold of dimension $n \geqslant 1$ and S^1 is the unit circle.

Our basic datum is a complex, real-analytic, closed 1-form ω defined on M. To ω we associate the line subbundle $T' \subset \mathbf{C} \otimes T^*(M \times S^1)$ spanned by the 1-form $dx - \omega$, where x denotes the angular variable in S^1. Its orthogonal $\mathscr{L} = (T')^{\perp} \subset \mathbf{C} \otimes T(M \times S^1)$ is then a vector subbundle of $\mathbf{C} \otimes T(M \times S^1)$ of fiber dimension n that can locally be described as follows: if $(V, t_1, ..., t_n)$ is a coordinate system on M such that $dh = \omega$ in V for some $h \in C^{\omega}(V)$, the pairwise commuting vector fields

$$L_j = \frac{\partial}{\partial t_j} + \frac{\partial h}{\partial t_j} \frac{\partial}{\partial x}, \qquad j = 1, ..., n \tag{1.1}$$

span $\mathscr{L}$ over $V \times S^1$. Thus $\mathscr{L}$ defines a *locally integrable structure of codimension one over* $M \times S^1$ [T2, T3].

To the structure $\mathscr{L}$ it is possible to associate, by means of a general construction, a complex of differential operators on $M \times S^1$ (see [T1], [T3]). The first stage of this complex is the differential operator $\mathbf{L}$: $C^{\infty}(M \times S^1) \to \Lambda^1 C^{\infty}(M \times S^1)$ given by

$$\mathbf{L} = d_t + \omega(t) \wedge \partial_x, \tag{1.2}$$

where d_t is the exterior derivative in M.

Our main purpose in this work is to present conditions for the operator $\mathbf{L}$ to be globally hypoelliptic, in the sense of the definition below.

Definition 1.1. The operator $\mathbf{L}$ is said to be *globally hypoelliptic* if

$$\mathscr{E} \doteq \{u \in \mathscr{D}'(M \times S^1) : \mathbf{L}u \in \Lambda^1 C^{\infty}(M \times S^1)\} \subset C^{\infty}(M \times S^1). \tag{1.3}$$

Recall that the characteristic set of $\mathscr{L}$ is the subset of $T^*(M \times S^1)$ given by

$$\mathrm{Char}(\mathscr{L}) = T' \cap (T^*(M \times S^1) \backslash 0). \tag{1.4}$$

From now on let us write $\omega = a + ib$, with a and b real, and $i = \sqrt{-1}$. The image of $\mathrm{Char}(\mathscr{L})$ under the natural projection $T^*(M \times S^1) \to M$ is the set

$$\Sigma = \{t \in M : b(t) = 0\}. \tag{1.5}$$

Hence $\mathscr{L}$ is elliptic in a neighborhood of each $(t, x) \notin \Sigma \times S^1$ and therefore, due to well-known results of local hypoellipticity, we just have to study the regularity of the elements of $\mathscr{E}$ near $\Sigma \times S^1$.

On the other hand, results of partial hypoellipticity imply that $\mathscr{E}$ is a subspace of $C^\infty(M, \mathscr{D}'(S^1))$ (see [T3, p. 25]).

Now, let $u \in \mathscr{E}$, $f = \mathbf{L}u$, and consider the following Fourier expansions in the x variable

$$u = \sum_{j=-\infty}^{+\infty} \hat{u}_j(t)\, e^{ijx}, \qquad f = \sum_{j=-\infty}^{+\infty} \hat{f}_j(t)\, e^{ijx}. \tag{1.6}$$

Note that, due to the remark above, $\hat{u}_j \in C^\infty(M)$, $\forall j \in \mathbf{Z}$.

LEMMA 1.2. *Let V be an open subset of M and $u \in \mathscr{E}$ such that*

$$\sup_{t \in V} \sup_{j \in \mathbf{Z}} |(1 + |j|)^N \hat{u}_j(t)| < \infty, \qquad \forall N \in \mathbf{Z}_+. \tag{1.7}$$

Then $u \in C^\infty(V \times S^1)$.

Proof. We can assume that V is the domain of a coordinate system $(t_1, ..., t_n)$. Note also that (1.7) remains valid if we replace u by $\partial_x^l u$, $l \in \mathbf{Z}_+$.

Since $u \in \mathscr{E}$, using the equation

$$d_t u = -\omega \wedge \partial_x u + f,$$

with $f \in \Lambda^1 C^\infty(M \times S^1)$, we conclude, by induction, that (1.7) is also valid when we replace u by $\partial_t^\alpha \partial_x^l u$, for any α and l. So $u \in C^\infty(V \times S^1)$. ∎

2. THE PURELY REAL CASE

In this section we assume M to be just C^∞ and study the global hypoellipticity of the operator

$$\mathbf{L} = d_t + a \wedge \partial_x, \tag{2.1}$$

where a is a closed, C^∞ 1-form in M which we assume to be real. In what follows we will adopt the following terminology: a "1-cycle in M" is an element in the free abelian group over the set of all smooth 1-chains in M which is in the kernel of the boundary operator.

DEFINITION 2.1. *For a closed, real* $\alpha \in \Lambda^1 C^\infty(M)$, we define:

(i) α is *integral* if

$$\frac{1}{2\pi} \int_\sigma \alpha \in \mathbf{Z}$$

for any 1-cycle σ in M.

(ii) α is *rational* if there exists $q \in \mathbb{N}$ such that $q\alpha$ is an integral 1-form.

(iii) α is *Liouville* if α is not rational and there are a sequence of closed integral 1-forms $\{\alpha_j\}$ and a sequence of integers $q_j \geqslant 2$ such that $\{q_j^j(\alpha - (1/q_j)\,\alpha_j)\}$ is bounded in $\Lambda^1 C^\infty(M)$.

Some remarks are in order. First, it is very easy to see that the concept of a 1-form being integral, rational or Liouville depends only on its cohomological class. The second remark is that in Definition 2.1(iii), we may assume that $q_j \to \infty$. Indeed, let σ be a 1-cycle in M such that $(1/2\pi) \int_\sigma \alpha$ is not a rational number. Since

$$\left| \int_\sigma \alpha - \frac{1}{q_j} \int_\sigma \alpha_j \right| \leqslant \frac{C}{q_j^j}$$

we have $(1/2\pi)(1/q_j) \int_\sigma \alpha_j \to (1/2\pi) \int_\sigma \alpha$ when $j \to \infty$ and therefore q_j cannot be bounded.

The next result sheds some light in the use of the terminology. Choose 1-cycles $\sigma_1, ..., \sigma_\nu$ such that their homological classes form a basis of the free part of $H_1(M, \mathbf{Z})$ and consider the linear map I: $H^1(M, \mathbf{R}) \to \mathbf{R}^\nu$ given by

$$I([\beta]) = \frac{1}{2\pi} \left(\int_{\sigma_1} \beta, ..., \int_{\sigma_\nu} \beta \right).$$

PROPOSITION 2.2. *Let* $\alpha \in \Lambda^1 C^\infty(M)$ *be real and closed. Then*

(i) *α is integral if and only if* $I([\alpha]) \in \mathbf{Z}^\nu$;

(ii) *α is rational if and only if* $I([\alpha]) \in \mathbf{Q}^\nu$;

(iii) *α is Liouville if and only if $I([\alpha]) \notin \mathbf{Q}^\nu$ and there are $p_j \in \mathbf{Z}^\nu$, $q_j \in \mathbf{Z}_+$, $q_j \geqslant 2$ and $C > 0$ such that, for all $j \in \mathbf{Z}_+$,*

$$\left| I([\alpha]) - \frac{1}{q_j} p_j \right| \leqslant \frac{C}{q_j^j}. \tag{2.2}$$

In particular, when $\nu = 1$ the form α is Liouville in the sense of Definition 2.1 if and only if $I([\alpha])$ is a Liouville number [H–W]. We also point out that vectors in $\mathbf{R}^\nu$ satisfying the approximation condition (2.2) have already appeared in [M1].

Proof. All the statements are very easy to check. The only possible exception is, perhaps, the proof that condition (iii) in Proposition 2.2 and condition (iii) in Definition 2.1 are equivalent. But this is a consequence of the following remark: if g is any Riemannian metric on M and if $H^1(M, g)$ denote the space of all harmonic 1-forms on M with respect to g, Hodge's Theorem shows that the map $H^1(M, g) \ni \beta \to I([\beta]) \in \mathbf{R}^\nu$ is a linear topological isomorphism when $H^1(M, g)$ is equipped with the C^∞ topology. Thus this map preserves bounded sets, proving the result. ∎

In the lemma below we use the notation $\Pi: \hat{M} \to M$ for the universal covering space of M.

LEMMA 2.3. *Let $\alpha \in \Lambda^1 C^\infty(M)$ be real and closed. Then α is integral if and only if given any $\psi \in C^\infty(\hat{M})$ such that $d\psi = \Pi^*(\alpha)$ the following holds:*

$$P, Q \in \hat{M}, \qquad \Pi(P) = \Pi(Q) \Rightarrow \psi(P) - \psi(Q) \in 2\pi\mathbf{Z}.$$

The proof follows easily from the fact that, according to Hurewicz Theorem, any 1-cycle in M is homologous to a smooth loop in M. The details are left to the reader.

We are now in position to prove the main result of this section:

THEOREM 2.4. *Let $a \in \Lambda^1 C^\infty(M)$ be real and closed. The operator* (2.1) *is globally hypoelliptic if and only if a is neither rational nor Liouville.*

Proof. First we assume that a is rational. Let $q \in \mathbb{N}$ be such that qa is integral and let $\psi \in C^\infty(\hat{M})$ with $d\psi = \Pi^*(qa)$. The previous lemma implies that $e^{i\psi} \in C^\infty(M)$. Define

$$u(t, x) = \sum_{N=1}^{\infty} e^{-iN(qx - \psi(t))}. \tag{2.3}$$

We have $u \in \mathscr{D}'(M \times S^1) \backslash C^\infty(M \times S^1)$ and $\mathbf{L}u = 0$ showing that $\mathbf{L}$ cannot be globally hypoelliptic in this case.

Now, we assume that a is Liouville. Let $\psi_j \in C^\infty(\hat{M})$ be such that $d\psi_j = \Pi^*(a_j)$, where $\{a_j\}$ is a sequence of closed integral forms such that $\{q_j^j(a - (1/q_j)\, a_j)\}$ is bounded in $\Lambda^1 C^\infty(M)$ for some sequence of integers $q_j \to \infty$. Set

$$u(t, x) = \sum_{j=1}^{\infty} e^{-i(q_j x - \psi_j(t))}. \tag{2.4}$$

We have $u \in \mathscr{D}'(M \times S^1) \backslash C^\infty(M \times S^1)$ and $\mathbf{L}u \in C^\infty(M \times S^1)$. In fact, a simple computation shows that

$$\mathbf{L}u = \sum_{j=1}^{\infty} iq_j \left(\frac{a_j}{q_j} - a\right) e^{-i(q_j x - \psi_j(t))}. \tag{2.5}$$

In order to prove that this series converges in C^∞ we first take a coordinate system $(V, t_1, ..., t_n)$ in M such that there is $\hat{V} \subset \hat{M}$ open which is mapped by Π diffeomorphically onto V.

Next we fix α, $\alpha' \in \mathbf{Z}_+^n$ and $l \in \mathbf{Z}_+$. There is $C > 0$ such that

$$\sup_V \left| \partial_t^{\alpha'} \left\{ \frac{a_j}{q_j} - a \right\} \right| \leqslant \frac{C}{q_j^j}, \qquad j \in \mathbf{Z}_+. \tag{2.6}$$

On the other hand there is $C_1 > 0$ such that

$$\sup_{V \times S^1} |\partial_x^l \partial_t^\alpha e^{-i(q_j x - \psi_j(t))}| \leqslant C_1 q_j^l \left(1 + \sum_{0 < \beta \leqslant \alpha} \sup_V |\partial_t^\beta (\psi_j \circ \Pi^{-1})| \right), \tag{2.7}$$

where Π^{-1} is the inverse of $\Pi\colon \hat{V} \to V$. Consequently, since $d(\psi_j \circ \Pi^{-1}) = a_j$ in V and $\{a_j/q_j\}$ is a bounded sequence in $\Lambda^1 C^\infty(M)$ we conclude the existence of $C_2 > 0$ such that

$$\sup_{V \times S^1} |\partial_x^l \partial_t^\alpha e^{-i(q_j x - \psi_j(t))}| \leqslant C_2 q_j^{l+1}, \qquad j \in \mathbf{Z}_+. \tag{2.8}$$

Then (2.6) and (2.8) show that (2.5) defines an element in $\Lambda^1 C^\infty(M)$ which concludes the proof that $\mathbf{L}$ is not globally hypoelliptic when a is Liouville.

It remains to show that $\mathbf{L}$ is globally hypoelliptic when a is neither rational nor Liouville.

Let $u \in \mathscr{E}$ and for $f = \mathbf{L}u$ we consider the Fourier series expansions given by (1.6); we must analyze the decay of $\{\hat{u}_j\}$ with respect to j. We have

$$d_t \hat{u}_j + ij\hat{u}_j a = \hat{f}_j. \tag{2.9}$$

Now choose $\psi \in C^\infty(\hat{M})$ such that $d\psi = \Pi^*(a)$, fix $t_0 \in M$ and select a coordinate neighborhood $(B, t_1, ..., t_n)$ of t_0 in M with B diffeomorphic to a ball and such that there is $\hat{B} \subset \hat{M}$ open so that $\Pi: \hat{B} \to B$ is a diffeomorphism. In B we have

$$d(e^{ij(\psi \circ \Pi^{-1})}\hat{u}_j) = e^{ij(\psi \circ \Pi^{-1})}\hat{f}_j \tag{2.10}$$

and thus, if we integrate over the segment $[t_0, t]$, we obtain

$$\hat{u}_j(t) = e^{ij[(\psi \circ \Pi^{-1})(t_0) - (\psi \circ \Pi^{-1})(t)]}\hat{u}_j(t_0) + e^{-i(\psi \circ \Pi^{-1})(t)} \int_{t_0}^{t} e^{ij(\psi \circ \Pi^{-1})}\hat{f}_j, \qquad t \in B. \tag{2.11}$$

In view of Lemma 1.2, since ψ is real and

$$\sup_{t \in B} \sup_{j \in \mathbf{Z}} |(1 + |j|)^N \hat{f}_j(t)| < \infty, \qquad N \in \mathbf{Z}_+$$

it will be sufficient to show that $\{\hat{u}_j(t_0)\}$ is a rapidly decaying sequence.

At this point we will make use of Proposition 2.2 and the notation there established. By Hurewicz Theorem we have the right to interpret $\sigma_1, ..., \sigma_\nu$ as smooth loops with base point t_0. We lift them as smooth curves $\tilde{\sigma}_k: [0, 1] \to \hat{M}$, such that $\tilde{\sigma}_k(0) = Q_0$, and set $\tilde{\sigma}_k(1) = Q_k$, $k = 1, ..., \nu$. If we now ntegrate the identity

$$d(e^{ij\psi}\Pi^*(\hat{u}_j)) = e^{ij\psi}\Pi^*(\hat{f}_j) \tag{2.12}$$

over $\tilde{\sigma}_k$, we obtain

$$\hat{u}_j(t_0) = e^{ij[\psi(Q_0) - \psi(Q_k)]}\hat{u}_j(t_0) + e^{-ij\psi(Q_k)} \int_{Q_0}^{Q_k} e^{ij\psi}\Pi^*(\hat{f}_j). \tag{2.13}$$

Therefore, if j and k are such that $j[\psi(Q_0) - \psi(Q_k)] \notin 2\pi\mathbf{Z}$, then

$$\hat{u}_j(t_0) = \frac{-e^{-ij\psi(Q_k)}}{e^{ij[\psi(Q_0) - \psi(Q_k)]} - 1} \int_{Q_0}^{Q_k} e^{ij\psi}\Pi^*(\hat{f}_j). \tag{2.14}$$

To prove that $\{\hat{u}_j(t_0)\}$ decays rapidly it suffices to show that

$$\max_k |e^{ij[\psi(Q_0) - \psi(Q_k)]} - 1| \geqslant C\,|j|^{-N} \tag{2.15}$$

for some $C > 0$, $N \in \mathbf{Z}_+$ and $\forall j \neq 0$.

Note that, for each $j \neq 0$, the set $\{k : 1 \leqslant k \leqslant \nu$ and $j[\psi(Q_0) - \psi(Q_k)] \notin 2\pi\mathbf{Z}\}$ is never empty, because a is not rational; hence, one has a fair chance of proving that (2.15) indeed holds. Since furthermore a is not Liouville, Proposition 2.2 implies the existence of constants $C>0$ and $L>0$ such that

$$\left|\frac{p}{q}-I([a])\right| \geqslant \frac{C}{q^L}, \tag{2.16}$$

for all $p \in \mathbf{Z}^\nu$ and all $q \in \mathbf{Z}_+$.

On the other hand, there is $\varepsilon_0>0$ such that for each $j \neq 0$ one of the conditions below occurs:

(i) For every $k \in \{1, ..., \nu\}$ there is $p_k \in \mathbf{Z}$ such that

$$|e^{ij[\psi(Q_0)-\psi(Q_k)]}-e^{i2\pi p_k}| \geqslant \tfrac{1}{2}\,|j[\psi(Q_0)-\psi(Q_k)]-2\pi p_k|. \tag{2.17}$$

(ii) There is $k \in \{1, ..., \nu\}$ such that

$$|e^{ij[\psi(Q_0)-\psi(Q_k)]}-1| \geqslant \varepsilon_0. \tag{2.18}$$

If (ii) occurs then (2.15) is valid for these j (with $N=0$), whereas if (i) occurs then, from (2.16),

$$\begin{aligned}\max_k |e^{ij[\psi(Q_0)-\psi(Q_k)]}-1| &\geqslant C \max_k |j[\psi(Q_0)-\psi(Q_k)]-2\pi p_k| \\ &= C \max_k \left| j \int_{\sigma_k} a + 2\pi p_k \right| \\ &\geqslant C'\,|j|^{-L+1} \end{aligned} \tag{2.19}$$

and so (2.15) holds. Since $\{\hat{f}_j\}$ decays rapidly, we obtain from (2.14) that $\{\hat{u}_j(t_0)\}$ also has the same property, which concludes the proof of Theorem 2.4. ∎

An immediate consequence of Theorem 2.4 is the Greenfield–Wallach [G–W] result mentioned in the Introduction: when $M=S^1$, $\mathbf{L}$ is globally hypoelliptic if and only if $I([a])$ is irrational and not a Liouville number.

3. The Existence of the Semi-global Primitive

Now, we return to the operator (1.2). As we have pointed out in Section 1, to study the regularity of an arbitrary element $u \in \mathscr{E}$ it is enough to analyze its behaviour near $\Sigma \times S^1$. For this it will be important to know that b has a primitive in a full neighborhood of Σ, a fact that we prove next.

We now assume that b is not zero, which implies that the set $\Sigma = \{t \in M : b(t) = 0\}$ is a proper analytic subset of M. We will exploit this property: first, Σ can decomposed into a finite, disjoint union

$$\Sigma = \bigcup_{j=1}^{m} S_j, \tag{3.1}$$

where each S_j is a connected, embedded C^ω submanifold of M. The other property we will use is the following consequence of the Bruhat–Whitney Theorem (see [Hi]): given any analytic subset $\mathscr{A} \subset \mathbf{R}^n$ and any point $x_0 \in \mathscr{A}$ there is $\varepsilon_0 > 0$ such that for any ε, $0 < \varepsilon \leqslant \varepsilon_0$, the set $\{x \in \mathscr{A} : |x - x_0| < \varepsilon\}$ is connected.

PROPOSITION 3.1. *There exist an open set U such that $\Sigma \subset U \subset M$ and a function $\phi \in C^\omega(U)$ with*

$$d\phi = b \qquad \text{in } U, \qquad \phi = 0 \qquad \text{on } \Sigma. \tag{3.2}$$

Thus, as a germ of an analytic function on Σ, ϕ is uniquely determined.

Proof. Starting with a basis of open subsets of M, consisting of open sets diffeomorphic to balls of $\mathbf{R}^n$, it is possible, due to the remarks above, to obtain two open coverings $\{B_1, B_2, ..., B_N\}$, $\{B_1', B_2', ..., B_N'\}$ of Σ such that

(a) $B_j \Subset B_j'$ for $j = 1, 2, ..., N$;

(b) $B_j' \cap \Sigma$ is nonempty and connected, for $j = 1, 2, ..., N$;

(c) $B_j' \cap B_k'$ is connected for all $j, k = 1, 2, ..., N$;

(d) For each $j = 1, 2, ..., N$, there exists $\phi_j \in C^\omega(B_j')$ such that $d\phi_j = b$ in B_j'.

We pull the equation $d\phi_j = b$ back to $S_k \cap B_j'$. Since $b = 0$ in S_k it follows that ϕ_j is locally constant on $S_k \cap B_j'$ and then, by (b), we may assume

(e) $\phi_j = 0$ on $\Sigma \cap B_j'$, for $j = 1, 2, ..., N$.

Let now

$$\Gamma = \{\{i, j\} : \bar{B}_i \cap \bar{B}_j \cap \Sigma = \varnothing\}$$

and

$$U = \bigcup_{i=1}^{N} B_i \Big\backslash \bigcup_{\{j,k\} \in \Gamma} (\bar{B}_j \cap \bar{B}_k).$$

Note that U is open and that $\Sigma \subset U$. To finish the proof it suffices to show that the function $\phi : U \to \mathbf{R}$, is given by

$$\phi(p) = \phi_j(p) \qquad \text{if} \quad p \in U \cap B_j,$$

is well defined.

Let $p \in U \cap B_j \cap B_k$; then $p \in B'_j \cap B'_k$. Now (d) implies $d(\phi_j - \phi_k) = 0$ on $B'_j \cap B'_k$ and this, together with (c), yields $\phi_j - \phi_k = c_{jk}$ (constant) on $B'_j \cap B'_k$. Since

$$\varnothing = \bar{B}_j \cap \bar{B}_k \cap \Sigma \subset B'_j \cap B'_k \cap \Sigma$$

there exists $q \in B'_j \cap B'_k$ such that $\phi_j(q) = \phi_k(q) = 0$ (we have used (e)). Hence $c_{jk} = 0$ and so ϕ is indeed well defined. ∎

4. The General Case-Beginning

We now begin to study the global hypoellipticity of the operator $\mathbf{L}$ given in (1.2) when b is not identically zero.

The analysis of the local hypoellipticity for $\mathbf{L}$ was already carried out by Maire [Ma], who obtained the following result: if $\mathscr{U} = V \times J$ is an open subset of $M \times S^1$ with V (resp. J) being an open subset of M (resp. S^1) and if $\phi \in C^{\omega}(V)$ is a primitive of b then L is hypoelliptic in $\mathscr{U}$ if ϕ is an open map. By using his techniques we push the conclusion a little further, obtaining at the same time a result of "propagation of singularities" for the solutions $u \in \mathscr{E}$ of $Lu = f$. We keep the notation of Section 3.

Let $t_0 \in \Sigma$ and consider a coordinate system $(B, t_1, ..., t_n)$ centered at t_0 with B diffeomorphic to a ball. Assume that $B \subset U$ and that $B \cap \Sigma$ is connected.

THEOREM 4.1. *Let $u \in \mathscr{E}$ and suppose that one of the following properties holds*:

(i) *ϕ is an open map at t_0.*

(ii) *there is $t_* B \cap \Sigma$ such that $u(t_*, \cdot) \in C^{\infty}(S^1)$.*

Then $(\{t_0\} \times S_1) \cap \operatorname{singsupp}(u) = \varnothing$.

To prove this theorem, we take $\psi \in C^{\omega}(B)$ such that $d\psi = a$. If $h = \psi + i\phi$, for $f = \mathbf{L}u$, $u \in \mathscr{E}$ we can write

$$d_t(e^{ijh(t)}\hat{u}_j(t)) = e^{ijh(t)}\hat{f}_j(t), \qquad t \in B. \tag{4.1}$$

We fix $t \in B$ and consider a piecewise C^1 curve γ_t: $[0, 1] \to B$, such that $\gamma_t(0) = t$. From (4.1), integrating along γ_t, we obtain

$$\hat{u}_j(t) = e^{-ij[h(t) - h(\gamma_t(1))]}\hat{u}_j(\gamma_t(1)) - \int_{\gamma_t} e^{-ij[h(t) - h(s)]}\hat{f}_j(s)\, ds. \tag{4.2}$$

If we can choose γ_t in a suitable fashion we can use Lemma 1.2 to prove the global hypoellipticity of the operator L. For this we need some preliminary results. The first one, due to Maire [Ma], gives, for a real-valued, smooth function satisfying the Lojasiewicz's inequality, some information on its variation along its gradient flow.

LEMMA 4.2. *Let $B \subset \mathbf{R}^n$ be an open ball centered at the origin and let $\Phi \in C^\infty(\bar{B})$ be a real-valued function satisfying $\Phi(0) = 0$, and such that*

$$|\Phi|^\theta \leqslant C\,|\nabla\Phi| \qquad \text{in } B, \tag{4.3}$$

where $C > 0$ and $\theta \in [0, 1)$.

Consider, for $t \in B$ with $\nabla\Phi(t) \neq 0$, the solution σ_t of

$$\sigma_t' = \frac{\nabla\Phi}{|\nabla\Phi|}(\sigma_t), \qquad \sigma_t(0) = t \tag{4.4}$$

defined in its maximal interval $[0, \delta(t))$. Then

$$\Phi(\sigma_t(\tau)) \geqslant \Phi(t) + C\tau^{1/(1-\theta)}, \qquad \tau \in [0, \delta(t)). \tag{4.5}$$

The next result is a consequence of more general results due to Hardt [Ha] and Tessier [Te] involving bounds for the length of curves joining points in analytic spaces.

LEMMA 4.3. *With the same notation as in Theorem 4.1, there is $K_1 > 0$ such that, given two arbitrary points in $B \cap \Sigma$, they can be joined by a piecewise analytic curve contained in $B \cap \Sigma$, whose length is less than or equal to K_1.*

We now present the key ingredient for the proof of Theorem 4.1.

LEMMA 4.4. *With the same notation as above, let $t_* \in B \cap \Sigma$. Then there exist*

(a) *an open neighborhood $B_0 \subset B$ of t_0;*

(b) *constants $K > 0$, $\varepsilon > 0$;*

(c) *a family $\{\gamma_t\}_{t \in B_0}$ of piecewise analytic curves γ_t: $[0, 1] \to B$, such that*

(I) $\gamma_t(0)=t$, $\forall t \in B_0$;

(II) $\phi(\gamma_t(\tau)) \geqslant \phi(t)$, $\forall t \in B_0$, $\forall \tau \in [0, 1]$;

(III) *The length* $l(\gamma_t)$ *of* γ_t *is less than or equal to* K, $\forall t \in B_0$;

(IV) *If* $t \in B_0$, *then one of the following properties holds*:

(IV)$_1$ $\gamma_t(1)=t_*$

(IV)$_2$ $\phi(\gamma_t(1)) \geqslant \phi(t)+\varepsilon$.

Proof. We can assume that B is an open ball centered at the origin in $\mathbf{R}^n$ and that (4.3) holds for $\Phi=\phi$ thanks to Lojasiewicz's inequality [L].

For each $t \in B \backslash \Sigma$, we take $\sigma_t(\tau)$, $\tau \in [0, \delta(t))$, as in Lemma 4.2. It follows from inequality (4.5) that

$$C\delta(t)^{1/(1-\theta)} \leqslant 2\, \|\phi\|_{L^\infty(B)}. \tag{4.6}$$

On the other hand, since $|\sigma_t(\tau)-\sigma_t(\tau')| \leqslant |\tau-\tau'|$, τ, $\tau' \in [0, \delta(t))$, (4.6) implies the existence of $\bar{t}=\lim_{\tau \nearrow \delta(t)} \sigma_t(\tau) \in \partial B \cup \Sigma$.

Let B_0 be another ball centered at the origin and denote $d=\operatorname{dist}(B_0, \partial B)$. We first assume $t \in B_0 \backslash \Sigma$. We need to consider two cases:

Case 1. $\bar{t} \in \partial B$.

We have

$$d \leqslant |\bar{t}-t| = |\sigma_t(\delta(t))-\sigma_t(0)| \leqslant \delta(t).$$

Since $\phi(\bar{t}) \geqslant \phi(t)+C\delta(t)^{1/(1-\theta)}$, taking $\varepsilon=Cd^{1/(1-\theta)}$ and $\gamma_t(\tau)=\sigma_t(\tau\delta(t))$, $\tau \in [0, 1]$, we see that (I), (II), and (IV)$_2$ hold.

To estimate the length, we define

$$K=\left[\frac{2}{C}\, \|\phi\|_{L^\infty(B)}\right]^{1-\theta}+K_1, \tag{4.7}$$

where K_1 is given by Lemma 4.3, and by (4.6),

$$l(\gamma_t)=\delta(t) \leqslant K.$$

Case 2. $\bar{t} \in \Sigma$.

In this case, since $\phi=0$ on Σ, we have $\phi(t) \leqslant 0$. Using Lemma 4.3, we can take a piecewise analytic curve η: $[\frac{1}{2}, 1] \to B$ contained in Σ, with $\eta(\frac{1}{2})=\bar{t}$, $\eta(1)=t_*$ and $l(\eta) \leqslant K_1$. We then define γ_t: $[0, 1] \to B$ by setting

$$\gamma_t(\tau)=\begin{cases}\sigma_t(2\tau\delta(t)), & \text{if } 0 \leqslant \tau \leqslant \frac{1}{2};\\ \eta(\tau), & \text{if } \frac{1}{2} \leqslant \tau \leqslant 1.\end{cases}$$

It immediately follows that (I), (II), (III), and $(\mathrm{IV})_1$ hold in this case, because $\phi(\eta(\tau))=0$, $\forall\tau\in[\frac{1}{2},1]$.

To finish the proof, it remains to examine the case when $t\in B_0\cap\Sigma$. But Lemma 4.3 implies the result: there exists a piecewise analytic curve $\gamma_t\colon[0,1]\to U$, $\gamma_t(0)=t$, $\gamma_t(1)=t_*$, with its image contained in $B\cap\Sigma$ such that $l(\gamma_t)\leqslant K_1\leqslant K$. Since $\phi\equiv 0$ in Σ, the proof is complete.

LEMMA 4.5. *Suppose that ϕ is an open function at the point t_0. Then, there exist*

(a) *an open neighborhood $B_0\subset B$ of t_0;*

(b) *constants $K>0$, $\varepsilon>0$;*

(c) *a family $\{\gamma_t\}_{t\in B_0}$ of piecewise analytic curves $\gamma_t\colon[0,1]\to B$, such that*

(I) $\gamma_t(0)=t$, $\forall t\in B_0$;

(II) $\phi(\gamma_t(\tau))\geqslant\phi(t)$, $\forall t\in B_0$, $\forall\tau\in[0,1]$;

(III) $l(\gamma_t)\leqslant K$, $\forall t\in B_0$;

(IV) $\phi(\gamma_t(1))\geqslant\phi(t)+\varepsilon$, $\forall t\in B_0$.

Proof. Since $t_0\in\overline{\{t\in B_0:\phi(t)>0\}}$, we can apply the Bruhat–Whitney Theorem [Hi] and consider an analytic curve $\lambda\colon[-\frac{1}{2},\frac{1}{2}]\to B_0$ such that $\lambda(0)=t_0$ and $\phi(\lambda(\tau))>0$ if $\tau\in(0,\frac{1}{2}]$.

Returning to Case 2 of the Lemma 4.4, taking $t_*=t_0$, considering a reparametrization of the curve

$$\gamma_t=\sigma_t+\eta+\lambda+\sigma_{\lambda(1/2)}$$

to the interval $[0,1]$ and redefining K, we conclude the proof. ∎

Remark 4.6. The proofs of Lemma 4.4 and Lemma 4.5 can be quickly modified to obtain a new family of piecewise analytic curves $\{\tilde\gamma_t\}_{t\in B_0}$, $\tilde\gamma_t\colon[0,1]\to B$, such that the following properties hold

$$\phi(\tilde\gamma_t(\tau))\leqslant\phi(t),\qquad \forall t\in B_0,\ \forall\tau\in[0,1]$$

$$\phi(\tilde\gamma_t(1))\leqslant\phi(t)\leqslant\phi(t)-\varepsilon,\qquad \forall t\in B_0$$

and the other conditions remain the same.

Proof of Theorem 4.1. We consider the family $\{\gamma_t\}_{t\in B_0}$ of curves described in Lemmas 4.4 and 4.5. According to them, there is a partition $B_0=B_{01}\cup B_{02}$ of B_0 such that:

$$\text{If } t\in B_{01} \text{ then } \gamma_t(1)=t_*; \tag{4.8}$$

$$\text{If } t\in B_{02} \text{ then } \phi(\gamma_t(1))\geqslant\phi(t)+\varepsilon. \tag{4.9}$$

According to Lemma 4.5, if condition (i) of Theorem (4.1) holds, we can take $B_{01} = \varnothing$. Now, using (4.2) for this family of curves, we obtain

$$|\hat{u}_j(t)| \leqslant e^{j[\phi(t) - \phi(\gamma_t(1))]} \, |\hat{u}_j(\gamma_t(1))| + \int_{\gamma_t} e^{j[\phi(t) - \phi(s)]} \, |\hat{f}_j(s)| \, |ds|, \qquad t \in B_0. \tag{4.10}$$

Since $f \in \Lambda^1 C^\infty(M \times S^1)$, for any $k \in \mathbf{Z}_+$ there is $C_k > 0$ such that

$$\sup_{t \in B} |\hat{f}_j(t)| \leqslant C_k(1 + |j|)^{-k}, \tag{4.11}$$

and since $u \in \mathscr{E}$, there are $\nu \in \mathbf{Z}_+$, $\tilde{C} > 0$ such that

$$\sup_{t \in B} |\hat{u}_j(t)| \leqslant \tilde{C}(1 + |j|)^{\nu}. \tag{4.12}$$

Hence, if j, $N \in \mathbf{Z}_+$ and $t \in B_0$,

$$|\hat{u}_j(t)| \leqslant e^{[\phi(t) - \phi(\gamma_t(1))]j} \, |\hat{u}_j(\gamma_t(1))| + C_N K(1 + j)^{-N} \tag{4.13}$$

since $\phi(t) \leqslant \phi(\gamma_t(\tau))$, $\tau \in [0, 1]$. It immediately follows that, if $t \in B_{02}$, then

$$|\hat{u}_j(t)| \leqslant e^{-\varepsilon j} \tilde{C}(1 + j)^{\nu} + C_N K(1 + j)^{-N}. \tag{4.14}$$

Now we suppose that $t \in B_{01}$. Since $u(t_*, \cdot) \in C^\infty(S^1)$ for every $N \in \mathbf{Z}_+$ there is a constant $C'_N > 0$ such that

$$|\hat{u}_j(t_*)| \leqslant C'_N(1 + |j|)^{-N}, \qquad j \in \mathbf{Z}. \tag{4.15}$$

and therefore, if $j \in \mathbf{Z}_+$,

$$|\hat{u}_j(t)| \leqslant (C'_N + C_N K)(1 + j)^{-N}, \qquad \forall t \in B_{01}. \tag{4.16}$$

For $j < 0$, we obtain similar conclusions to (4.14) and (4.16), using the family $\{\tilde{\gamma}_t\}_{t \in B_0}$ given in Remark 4.6. Finally, Lemma 1.2 implies the result. ∎

COROLLARY 4.7. *Let u belong to $\mathscr{E}$ and suppose that for $\bar{t} \in \Sigma$ we have $(\{\bar{t}\} \times S^1) \cap \operatorname{singsupp}(u) = \varnothing$. If Σ_0 denotes the connected component of Σ such that $\bar{t} \in \Sigma_0$, then*

$$(\Sigma_0 \times S^1) \cap \operatorname{singsupp}(u) = \varnothing.$$

Proof. The set of all $t \in \Sigma_0$ such that $(\{t\} \times S^1) \cap \operatorname{singsupp}(u) = \varnothing$ is clearly open and by Theorem 4.1 it is closed. ∎

COROLLARY 4.8. *Let t_0 belong to Σ and suppose that ϕ is an open map at t_0. If Σ_0 denotes the connected component of Σ such that $t_0 \in \Sigma_0$, and $u \in \mathscr{E}$, then*

$$(\Sigma_0 \times S^1) \cap \operatorname{singsupp}(u) = \varnothing.$$

In particular, if all components of Σ have a point at which ϕ is an open map then the operator $\mathbf{L}$ *is globally hypoelliptic.*

5. THE GENERAL CASE: CONCLUSION

In this final section we study the case when, in a given component $\Sigma_0 \subset \Sigma$, the function ϕ is not open at any of its points. Since ϕ is analytic, this is equivalent to saying that ϕ has fixed sign in some open neighborhood $U_0 \subset U$ of Σ_0. Since then $\phi = 0$ implies $d\phi = 0$ we can restrict ourselves to the following situation:

$$\text{Either } \phi > 0 \text{ or } \phi < 0 \text{ in } U_0 \backslash \Sigma_0. \tag{5.1}$$

PROPOSITION 5.1. *Suppose that* (5.1) *holds and that a has a primitive in some open neighborhood $V_0 \subset U_0$ of Σ_0. Then* $\mathbf{L}$ *is not globally hypoelliptic.*

Proof. Assume $\phi > 0$ in $U_0 \backslash \Sigma_0$, the other case being analogous. Let $\psi \in C^\omega(V_0)$ be such that $d\psi = a$. The function $v(x, t) = \exp(i(\psi(t) + i\phi(t) - x))$ satisfies $\mathbf{L}v = 0$ in $V_0 \times S^\perp$, $|v(x, t)| \leqslant 1$. We then define $v^\flat(x, t) = (1 - v(x, t))^{1/2}$, where the branch of the function $\zeta \to (1 - \zeta)^{1/2}$ is defined in the complex plane after removing the ray $\{x \in \mathbf{R} : x \geqslant 1\}$. Hence $v^\flat$ is continuous, satisfies $\mathbf{L}v^\flat = 0$ and its singular support intersects $\{t\} \times S^1$ for every $t \in \Sigma_0$. Finally we select $\chi \in C_c^\infty(V_0)$ such that $\chi \equiv 1$ in a neighborhood of Σ_0 and set $u = \chi v^\flat$. We have $u \in \mathscr{E} \backslash C^\infty(M \times S^1)$, which concludes the proof. ∎

COROLLARY 5.2. *Assume that the cohomological class of a vanishes near Σ. Then* $\mathbf{L}$ *is globally hypoelliptic if and only if every connected component of Σ has a point at which ϕ is an open map.*

It is worth recalling that the hypotheses of Proposition 5.1 are satisfied when Σ_0 reduces to a single point. In general, however, when the cohomological class of a does not vanish near Σ_0, a different phenomenon regarding the regularity of the elements of $\mathscr{E}$ near Σ_0 may occur. In order to describe it we make now the strong hypothesis that Σ_0 is an analytic, embedded submanifold of M of dimension $\geqslant 1$.

The locally integrable structure $\mathscr{L}$ induces a (real) locally integrable structure $\mathscr{L}_0$ over $\Sigma_0 \times S^1$ whose orthogonal $\mathscr{L}_0^\perp \subset \mathbf{C} \otimes T^*(\Sigma_0 \times S^1)$ is spanned by the 1-form $dx - \iota^*(\omega) = dx - \iota^*(a)$, where ι is the inclusion map $\Sigma_0 \subsetneq M$. The operator $\mathbf{L}_0 : C^\infty(\Sigma_0 \times S^1) \to \Lambda^1 C^\infty(\Sigma_0 \times S^1)$ associated to $\mathscr{L}_0$ is given by

$$\mathbf{L}_0 = d_0 + \iota^*(a) \wedge \partial_x, \tag{5.2}$$

where d_0 denotes the exterior derivative on Σ_0.

THEOREM 5.3. *Suppose that* (5.1) *holds and that* Σ_0 *is an embedded submanifold of* M *of dimension* $\geqslant 1$. *The following statements are equivalent*:

(i) $\operatorname{singsupp}(u) \cap (\Sigma_0 \times S^1) = \varnothing$, $\forall u \in \mathscr{E}$;

(ii) $\mathbf{L}_0$ *is globally hypoelliptic*;

(iii) $\iota^*(a) \in \Lambda^1 C^\infty(\Sigma_0)$ *is neither rational nor Liouville.*

Proof. Theorem 2.4 shows that (ii) and (iii) are equivalent. Now assume (ii) and take $u \in \mathscr{E}$. Since $u \in C^\infty(M, \mathscr{D}'(S^1))$ we can restrict it to Σ_0 and thus obtain an element $u_0 \in C^\infty(\Sigma_0, \mathscr{D}'(S^1))$ such that $\mathbf{L}_0 u_0 \in \Lambda^1 C^\infty(\Sigma_0 \times S^1)$. By (ii) $u_0 \in C^\infty(\Sigma_0 \times S^1)$ and then $\operatorname{singsupp}(u) \cap (\Sigma_0 \times S^1) = \varnothing$ by Theorem 4.1.

It remains to prove that (i) implies (iii). Here we again assume that $\phi > 0$ in $U_0 \backslash \Sigma_0$, the other case requiring minor modifications in the argument. We need some preparatory results. By constructing a tubular neighborhood around Σ_0 in M we obtain the existence of an open neighborhood $W_0 \subset U_0$ of Σ_0 in M and an analytic retraction $\rho: W_0 \to \Sigma_0$ such that $\iota \circ \rho: W_0 \to W_0$ is homotopic to the identity map $W_0 \to W_0$. Thus $\iota \circ \rho$ induces the identity homomorphism in $H^1(W_0, \mathbf{R})$ and consequently we can insure the existence of $\eta \in C^\infty(W_0)$ real such that

$$a = \rho^*(\iota^*(a)) + d\eta \qquad \text{in } W_0. \tag{5.3}$$

Next let $v_0 \in C^\infty(\Sigma_0, \mathscr{D}'(S^1))$ have Fourier expansion

$$v_0 = \sum_{j=0}^{\infty} v_{0_j} e^{-ijx} \tag{5.4}$$

and define

$$v(t, x) = \sum_{j=0}^{\infty} \rho^*(v_{0_j})(t)\, e^{-ij(x - \eta(t) - i\phi(t))}. \tag{5.5}$$

Since $|e^{-ij(x-\eta(t)-i\phi(t))}| = e^{-j\phi(t)} \leqslant 1$ if $(t, x) \in W_0 \times S^1$ and $j \geqslant 0$ it is easily seen that $v \in C^\infty(W_0, \mathscr{D}'(S^1))$. Moreover, if we write $f_0 \doteq \mathbf{L}_0 v_0 = \sum_{j=0}^\infty f_{0_j} e^{-ijx}$ a very simple computation, which makes use of (5.3), gives

$$\mathbf{L}v(t, x) = \sum_{j=0}^{\infty} \rho^*(f_{0_j})\, e^{-ij(x-\eta(t)-i\phi(t))}, \qquad (t, x) \in W_0 \times S^1. \tag{5.6}$$

Finally assume that (iii) does not hold. The proof of Theorem 2.4 gives then a nonsmooth distribution v_0 in $\Sigma_0 \times S^1$ as in (5.4) such that $\mathbf{L}_0 v_0 \in \Lambda^1 C^\infty(\Sigma_0 \times S^1)$. Then v given by (5.5) is not smooth but, by (5.6), $\mathbf{L}v \in \Lambda^1 C^\infty(W_0 \times S^1)$. Taking a cut-off function $\chi \in C_c^\infty(W_0)$ such that $\chi \equiv 1$ in a neighborhood of Σ_0 and defining $u = \chi v$ it follows that $u \in \mathscr{E}$ but $\operatorname{singsupp}(u) \cap (\Sigma_0 \times S^1) \neq \varnothing$, showing that (i) cannot hold. This completes the proof of Theorem 5.3. ∎

COROLLARY 5.4. *Assume that all connected components Σ' of Σ are embedded analytic submanifolds of M. The operator* $\mathbf{L}$ *is globally hypoelliptic if and only if for each Σ' one of the conditions below holds true*:

a. *There is $t' \in \Sigma'$ such that ϕ is open at t'*;

b. *Σ' has dimension $\geqslant 1$, the function ϕ has fixed sign in a neighborhood of Σ' and if ι' denotes the inclusion $\Sigma' \hookrightarrow M$ the real form $\iota'^*(a) \in \Lambda^1 C^\omega(\Sigma')$ is neither rational nor Liouville.*

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